Linear Analysis of Liquid Film Flow

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Analyses of the flow of thin Newtonian liquid films over vertical plane surfaces at low Reynolds number in the wavy flow regime have been considered. The results of a linear analysis of the equations of motion, seeking a steady periodic solution to characterize the wave profile, are presented. The solutions derived in this work give better agreement with experimental results than previously published solutions using a linear analysis. Predictions of the wave properties (wave celerity, wave number, and wavelength) as functions of the flow conditions are derived from the theory.

The flow of thin liquid films over inclined plane surfaces has been studied by many investigators since Nusselt's classical analysis (*14*) presented in 1916. An understanding of the flow phenomena is important since liquid films are utilized in many chemical engineering operations, for example in heat and mass transfer equipment and chemical reactors.

The plane flow solution (*14*) is not valid at higher flow rates when gravity and capillary waves are generated at the free surface. Kirkbride (*11*) was one of the first workers to recognize these waves, and showed that the surface waves were responsible for the observed deviations from Nusselt's analysis. Experimental investigations and techniques adopted to determine these characteristics have been reviewed adequately by Fulford (*4*) and Rushton (*15*).

Theoretical researchers have attempted to analyze the behavior of wavy film flow by two approaches: (1) by searching for steady periodic solutions of the equations of motion (*8, 3, 12, 10, 13*); and (2) by application of the theory of hydrodynamic stability (*2, 19, 5, 7, 17*). A summary of some of the main results of these theoretical treatments appears in references *4, 15,* and *13*.

In the present paper solutions are presented using the first technique. The Navier-Stokes equations and the boundary conditions are simplified by application of an order of magnitude analysis. Periodic solutions of the approximate equations of motion are obtained by averaging the equations over the film thickness.

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Before a detailed analysis is presented, it is necessary to establish a consistent procedure to be adopted in formulating the equations of motion and the boundary conditions. Some discrepancies between previous analyses have arisen in the simplification of the equations (*3, 8, 12, 13*).

The equations of motion and continuity for two-dimensional vertical flow of an incompressible fluid are

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \left\{ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right\} + g - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (1)$$

$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = \nu \left\{ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right\} - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \quad (3)$$

the coordinate system being illustrated in Figure 1.

In order to simplify these equations for thin film flow it is necessary to consider the magnitudes of the various terms. To do this, it is convenient to render the equations dimensionless prior to the analysis. Since a periodic solution is required for wave flow, a convenient scaling parameter in the x direction along the flow is the wavelength λ and for the y direction the mean film thickness h_0. Over one period x has a range λ, from $x = x_0$ to $x = x_0 + \lambda$.

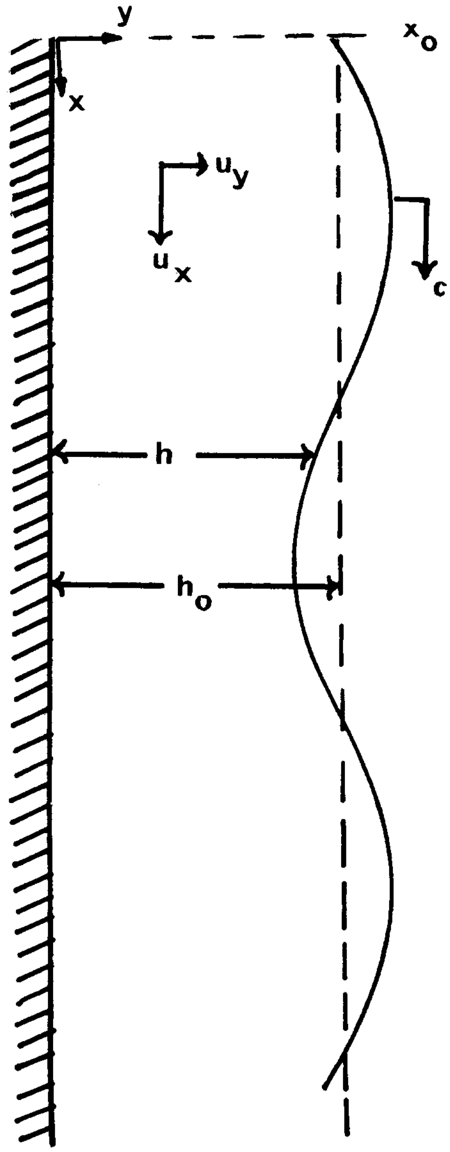

Fig. 1. Coordinate system for wavy laminar film flow.

Thus a dimensionless distance X may be defined by

$$X = \frac{x - x_0}{\lambda}$$

hence $0 < X < 1$ and $\partial/\partial X = 0(1)$. Similarly for the y direction let $Y = y/h_0$; then, for small surface displacements, $\partial/\partial Y = 0(1)$. In a similar manner nondimensional velocity components U, V of order unity may be defined by $U = u_x/u_0$ and $V = (u_y\lambda)/(u_0 h_0)$, where u_0 is the velocity at the mean stream section h_0. The continuity Equation (3) then becomes

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad (4)$$

In addition it is assumed that the liquid film has a wave motion propagating in the x direction with constant wave celerity c. Hence solutions for the velocity, pressure, and film thickness periodic in x and t may be expressed in terms of the single variable $x - ct$. Thus dimensionless quantities for time, pressure, and wave celerity may be defined by

$$T = \frac{\zeta u_0 t}{h_0}, \quad P = \frac{p}{\rho u_0^2}, \quad \alpha = \frac{c}{u_0}$$

where $\zeta = h_0/\lambda$ is a thin film parameter and P and α are considered of order unity. Then since $\partial/\partial t = -c\ \partial/\partial x$ using dimensionless terms $\partial/\partial T = -\alpha\ \partial/\partial X$, which implies that $\partial/\partial T$ is also of order unity.

The equations of motion may now be written in the dimensionless forms:

$$-\alpha\zeta\frac{\partial U}{\partial X} + \zeta U\frac{\partial U}{\partial X} + \zeta V\frac{\partial U}{\partial Y} = \frac{\nu}{h_0 u_0}\left\{\zeta^2\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right\} + \frac{g h_0}{u_0^2} - \zeta\frac{\partial P}{\partial X} \qquad (5)$$

$$-\alpha\zeta^2\frac{\partial V}{\partial X} + \zeta^2 U\frac{\partial V}{\partial X} + \zeta^2 V\frac{\partial V}{\partial Y} = \frac{\nu}{h_0 u_0}\left\{\zeta^3\frac{\partial^2 V}{\partial X^2} + \zeta\frac{\partial^2 V}{\partial Y^2}\right\} - \frac{\partial P}{\partial Y} \qquad (6)$$

The boundary conditions for the solution of this problem are

1. $u_x = u_y = 0$ at $y = 0$ for all values of x and t (no slip at the solid boundary) or

$$U = V = 0 \quad \text{at} \quad Y = 0 \qquad (7)$$

2. The free surface kinematic boundary condition

$$\frac{\partial h}{\partial t} + u_x\frac{\partial h}{\partial x} = u_y \ \text{at}\ y = h \quad \text{or}$$

$$\zeta\frac{\partial H}{\partial T} + \zeta U\frac{\partial X}{\partial X} = \zeta V \quad \text{at} \quad Y = H = \frac{h}{h_0} \qquad (8)$$

3. The stress conditions at the free surface become, on introducing the continuity equation (*1*)

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 2\frac{\partial u_x}{\partial x}\tan 2\delta \quad \text{at} \quad y = h$$

$$p - p_0 = p_\sigma + \frac{2\nu\rho}{\cos 2\delta}\frac{\partial u_y}{\partial y} \quad \text{at} \quad y = h$$

where $\tan\delta = \partial h/\partial x$, p_0 is the ambient pressure which may be taken as zero, and p_σ, the capillary pressure due to the curvature of the surface, is given by

$$p_\sigma = -\sigma\frac{\partial^2 h}{\partial x^2} \Bigg/ \left\{1 + \left(\frac{\partial h}{\partial x}\right)^2\right\}^{3/2}$$

In dimensionless forms boundary conditions 3 become

$$\frac{\partial U}{\partial Y} + \zeta^2\frac{\partial V}{\partial X} = 4\zeta^2\frac{\partial H}{\partial X}\frac{\partial U}{\partial X} \Bigg/ \left\{1 - \zeta^2\left(\frac{\partial H}{\partial X}\right)^2\right\} = 0\ (\zeta^2) \quad \text{at} \quad Y = H \qquad (9)$$

and

$$-P + \frac{8}{N_{Re}}\zeta\frac{\partial V}{\partial Y} - \frac{\zeta^2}{N_{We}}\frac{\partial^2 H}{\partial X^2} = 0\left\{\frac{\zeta^3}{N_{Re}}, \frac{\zeta^4}{N_{We}}\right\} \quad \text{at} \quad Y = H \qquad (10)$$

The terms on the right-hand side of Equations (9) and (10) represent the nonlinear contributions. N_{Re} and N_{We} are the Reynolds and Weber numbers. These are defined, together with the Froude number which appears in the equation of motion [Equation (5)], by

$$N_{Re} = \frac{4h_0 u_0}{\nu}, \quad N_{Fr} = \frac{u_0}{\sqrt{g h_0}}, \quad \text{and} \quad N_{We} = \frac{h_0 u_0^2 \rho}{\sigma}$$

The parameters N_{Re} and N_{Fr} will be assumed to be of order unity, and to retain the contribution due to surface tension forces in the analysis N_{We} is of order ζ [Equation (10)]. The long wave approximation $\zeta << 1$ now provides a means of ordering the terms of the governing equations of motion and boundary conditions

FIRST APPROXIMATION

Considering first, terms of order unity in Equations (4) to (6)

$$\frac{4}{N_{Re}}\frac{\partial^2 U}{\partial Y^2} + \frac{1}{N_{Fr}^2} = 0 \qquad (12)$$

$$\frac{\partial P}{\partial Y} = 0 \qquad (13)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad (14)$$

The corresponding boundary conditions are

$$U = V = 0 \quad \text{at} \quad Y = 0 \qquad (15a)$$

$$\frac{\partial U}{\partial Y} = 0 \quad \text{at} \quad Y = H \qquad (15b)$$

$$P = 0 \quad \text{at} \quad Y = H \qquad (15c)$$

$$\frac{\partial H}{\partial T} + U\frac{\partial H}{\partial X} = V \quad \text{at} \quad Y = H \qquad (15d)$$

The solution of Equations (12) and (14) is

$$U = 3\overline{U}_1\left\{\frac{Y}{H} - \frac{1}{2}\left(\frac{Y}{H}\right)^2\right\} \qquad (16)$$

$$V = -\frac{3}{2}\overline{U}_1\frac{\partial H}{\partial X}\left(\frac{Y}{H}\right)^2 \qquad (17)$$

where

$$\overline{U}_1 = \frac{1}{H}\int_0^H U\,dY = \frac{N_{Re}}{12 N_{Fr}^2}$$

The dimensionless film thickness H may be expressed in terms of a new variable $\varphi(X, T)$, having a zero mean value; thus

$$H = 1 + \varphi \qquad (18)$$

The relationship between the two dependent variables H

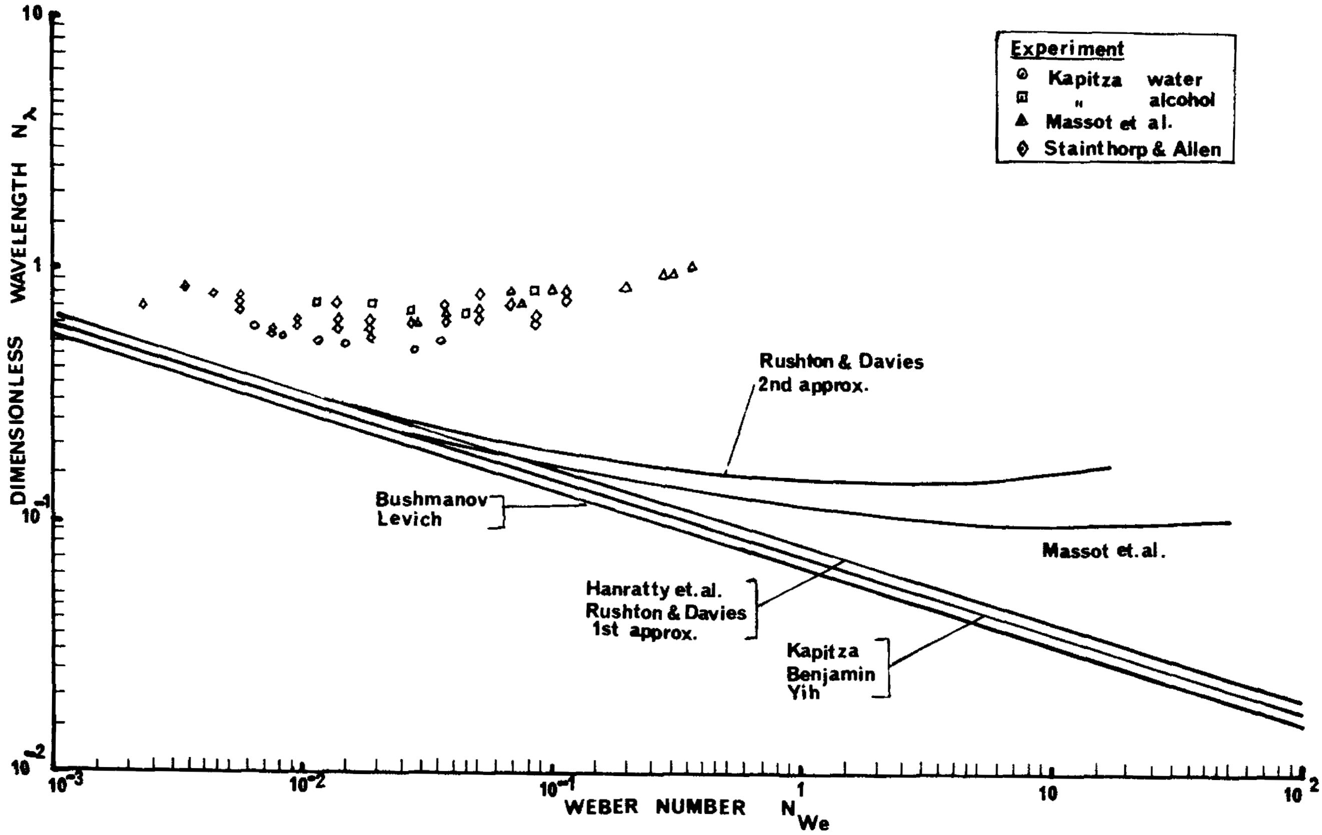

Fig. 2. Dimensionless wavelength vs. Weber number—a comparison between theory and experiment.

and $\overline{U}_1$ is provided by a macroscopic mass balance; thus

$$\frac{\partial H}{\partial T} = -\alpha \frac{\partial H}{\partial X} = -\frac{\partial}{\partial X}(\overline{U}_1 H) \tag{19}$$

Substitution of Equation (18) into this equation and integration from $X = 0$ to $X = 1$ gives

$$\overline{U}_1 = \frac{1 + \alpha\varphi}{1 + \varphi} \tag{20}$$

For solution, a second ordering parameter φ, representing the condition for small amplitude waves, must be considered. Substitution of Equations (16) and (17) into the kinematic condition [Equation (15*d*)] gives

$$\alpha = 3\overline{U}_1 \tag{21}$$

As $\varphi \to 0$ this solution tends to the Nusselt solution for plane flow and predicts that the wave celerity α approaches 3, in agreement with other workers (*8, 3, 12, 2, 19*).

SECOND APPROXIMATION

This approach can now be extended to include terms of order ζ in Equations (5) and (6). Thus these reduce to

$$\zeta\frac{\partial U}{\partial T} + \zeta U\frac{\partial U}{\partial X} + \zeta V\frac{\partial U}{\partial Y} = \frac{4}{N_{Re}}\frac{\partial^2 U}{\partial Y^2} + \frac{1}{N_{Fr}^2} - \zeta\frac{\partial P}{\partial X} \tag{22}$$

$$0 = \zeta\frac{4}{N_{Re}}\frac{\partial^2 V}{\partial Y^2} - \frac{\partial P}{\partial Y} \tag{23}$$

with corresponding boundary conditions

$$U = V = 0 \quad \text{at} \quad Y = 0 \tag{24a}$$

$$\frac{\partial U}{\partial Y} = 0 \quad \text{at} \quad Y = H \tag{24b}$$

$$-P + \frac{8}{N_{Re}}\zeta\frac{\partial V}{\partial Y} - \left(\frac{\zeta^2}{N_{We}}\right)\frac{\partial^2 H}{\partial X^2} = 0 \quad \text{at} \quad Y = H \tag{24c}$$

$$\frac{\partial H}{\partial T} + U\frac{\partial H}{\partial X} = V \quad \text{at} \quad Y = H \tag{24d}$$

In order to solve these equations it is necessary to assume a velocity distribution satisfying at least two of the prescribed boundary conditions. Kapitza postulated an approximate velocity profile of the Nusselt type

$$U = 3\,\overline{U}_2(X, T)\left\{\frac{Y}{H} - \frac{1}{2}\left(\frac{Y}{H}\right)^2\right\} \tag{25}$$

$\overline{U}_2$ being the mean velocity of the flow defined by

$$\overline{U}_2 = \frac{1}{H}\int_0^H U\,dY \tag{26}$$

This expression for U, reduces to the Nusselt solution for H constant ($= 1$), satisfies the no-slip condition at the solid boundary and the shear stress condition at the free surface. With the use of the continuity equation and Equation (25), the Y component of velocity is (*13*)

$$V = -\frac{3}{H}\frac{\partial \overline{U}_2}{\partial X}\left\{\frac{1}{2}Y^2 - \frac{1}{6}\frac{Y^3}{H}\right\} + \frac{3\overline{U}_2}{H^2}\frac{\partial H}{\partial X}\left\{\frac{1}{2}Y^2 - \frac{1}{3}\frac{Y^3}{H}\right\} \tag{27}$$

Hence integrating Equation (23) with respect to Y, using boundary condition (24*c*) the pressure distribution P may be determined as a function of X and Y. Introducing the velocity components U and V into the reduced equation of motion (22) and averaging this equation over the film thickness by integrating with respect to Y, one can obtain a third-order nonlinear differential equation relating H and $\overline{U}_2$. Hence introducing the expression for H and $\overline{U}_2$ in terms of the dimensionless displacement parameter φ and neglecting quadratic and higher powers of φ (assumed small since $\varphi << 1$), one obtains the following linearized equation of motion:

$$\frac{N_{Re}}{12N_{Fr}^2}\frac{N_{Fr}^2}{N_{We}}\zeta^3\varphi''' + \frac{1}{6}\zeta^2(5\alpha - 6)\varphi'' + \frac{N_{Re}}{12N_{Fr}^2}N_{Fr}^2\zeta\left(\alpha^2 - \frac{12\alpha}{5} + \frac{6}{5}\right)\varphi' - (\alpha - 3)\varphi + \left\{\frac{N_{Re}}{12N_{Fr}^2} - 1\right\} = 0 \quad (28)$$

The prime denotes differentiation with respect to $(X - \alpha T)$. Since the mean value of φ is zero, the constant in Equation (28) must vanish; thus

$$N_{Re} = 12N_{Fr}^2$$

Therefore the mean film thickness H_0 is constant and equal to the value obtained by Nusselt.

For a constant periodic solution of Equation (28) satisfying the boundary conditions $\overline{\varphi} = 0$ and $\varphi = 0$ at $X - \alpha T = 0$, assume

$$\varphi = \theta \sin n(x - ct)$$

where θ is an arbitrary constant ($<< 1$) representing the amplitude of the oscillation and n is the frequency of this oscillation. Introducing Reynolds, Froude, and Weber numbers and a dimensionless wave number, defined by $N_W = nh_0$, the supplementary conditions consistent with the periodic solution

$$\varphi = \theta \sin \frac{N_W}{\zeta}(X - \alpha T) \quad (29)$$

are

$$N_{Re} = 12N_{Fr}^2 \quad (30a)$$

$$N_W^2 = N_{We}\left(\alpha^2 - \frac{12\alpha}{5} + \frac{6}{5}\right) \quad (30b)$$

$$N_{We} = \frac{6(3-\alpha)}{(5\alpha - 6)\left(\alpha^2 - \frac{12\alpha}{5} + \frac{6}{5}\right)} \quad (30c)$$

This solution provides predictions for the film thickness, wave celerity, and wave number as functions of the flow parameters. The solution is in complete agreement with the instability analysis of Hanratty and Hershman (5) for the limiting case of low Weber numbers.

Conditions (30b) and (30c) yield the following relation:

$$N_W^2 = \frac{6(3-\alpha)}{(5\alpha - 6)}$$

which for long waves may be expanded in the form

$$\alpha = 3 - \frac{3}{2}N_W^2 + 0\,(N_W^4) \quad (31)$$

where

$$N_W^2 = 3N_{We} + 0(\zeta^2) \quad (32)$$

Thus this solution reduces to $\alpha = 3$ for plane flow in agreement with the various low Weber number analyses (2, 8, 19). By application of the theory of hydrodynamic stability, Benjamin (2) obtained an improved relation for the wave celerity.

$$\alpha = 3 - 3N_W^2 + 0.023(N_W R)^2 + 0(N_W^4) \quad (33)$$

where

$$R = N_{Re}/4$$

and

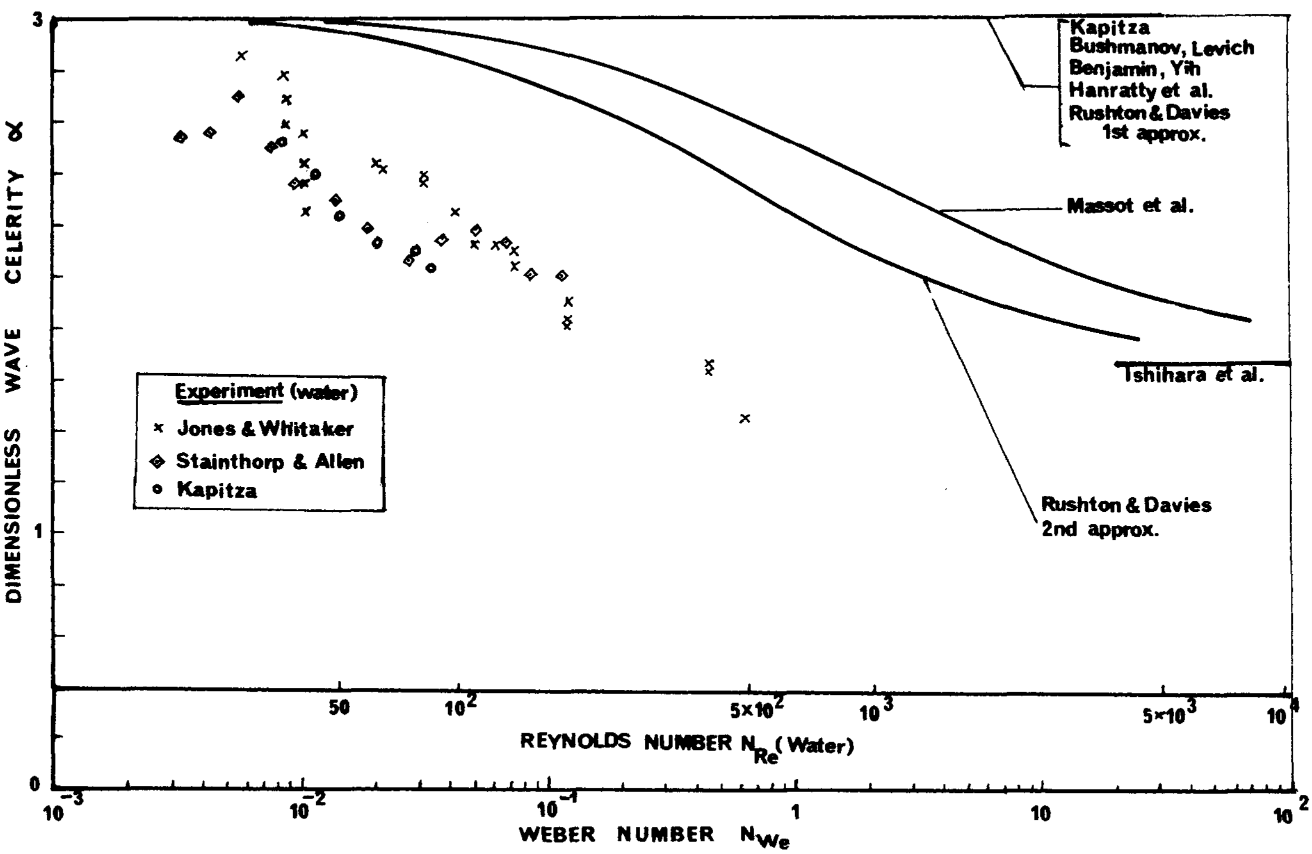

Fig. 3. Dimensionless wave celerity vs. Weber number—a comparison between theory and experiment.

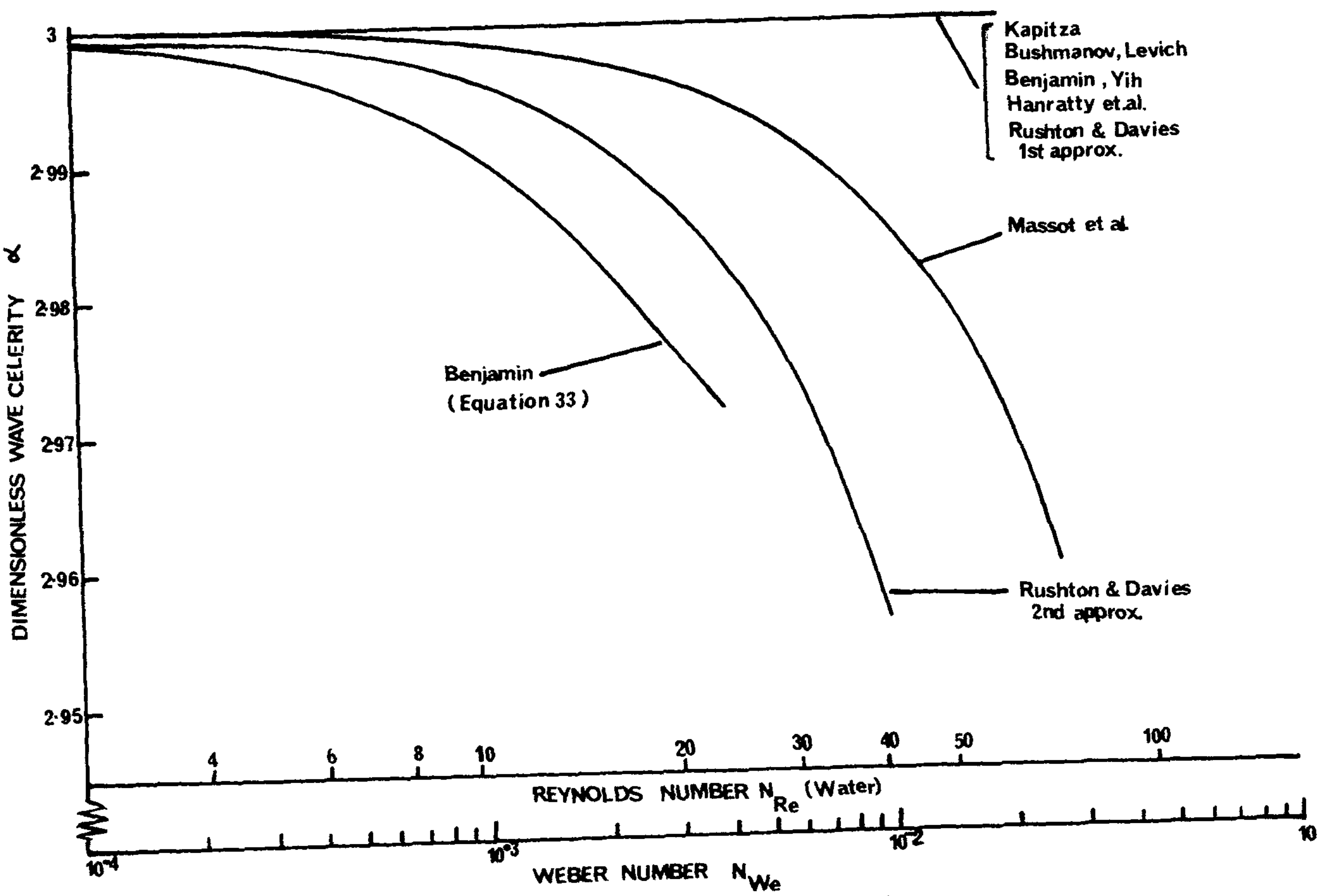

Fig. 4. Dimensionless wave celerity vs. Weber number.

$$N_W^2 = 3.6N_{We} + 0(N_{We}^3) \quad (34)$$

predicting that the wave celerity is a function of both the Weber number and the fluid properties group $N_{fp} = (3\gamma^2/g\nu^4)^{1/5}$, relating the Reynolds and Weber numbers by the expression

$$N_{Re} = 4N_{fp}\ N_{We}^{3/5}$$

This is in contradiction with the present solution; however over the regime of validity of the theory this effect is negligible. Defining a dimensionless wavelength N_λ

$$N_\lambda = \frac{\lambda}{2\pi}\sqrt{\frac{g}{\gamma}}$$

then

$$N_\lambda = \frac{\sqrt{12}}{\left(\alpha^2 - \frac{12\alpha}{5} + \frac{6}{5}\right)} N_{Re}^{-1/2} \quad (35)$$

Hence from the present treatment the Weber number and the dimensionless fluid properties group emerge as the dominant factors controlling wave properties. The postulated velocity profile, Equations (25) and (27), enables expressions for the surface displacement, wave celerity, wave number, and wavelength to be determined to order ζ. To extend this technique to include higher orders of ζ is however not possible since this approximate velocity profile does not satisfy the boundary condition for the rate of shearing at the free surface.

In this work a periodic solution satisfying simultaneously reduced forms of the equations of motion has been determined. Massot, Irani, and Lightfoot attempted to find a steady state periodic solution of the complete equations of motion. The main difference between their analysis and the present treatment appears in the formulation of the corresponding boundary conditions. They used simplified boundary conditions for the normal and shear stresses at the free surface and assumed the boundary layer approximation of hydrostatic equilibrium across the film. With these assumptions they obtained a pair of linear third-order differential equations in φ but were unable to find a periodic solution satisfying simultaneously both these equations. As a result they considered the contribution made by the x component of the equations of motion. Massot et al. were unable to find a unique solution since the equations of motion and boundary conditions were not consistent as is demonstrated in the present formulation. Comparison of the predictions of the present analyses with the existing theoretical analyses and experimental results will now be discussed.

Wavelength

The limiting form of Equation (35), as the Weber number tends to zero, is $N_\lambda = 2N_{Re}^{-1/2}$ which is the first approximation. The predicted variation of N_λ for water at 20°C. is shown in Figure 2.

The wavelength passes through a minimum and becomes very large for both large and small Weber numbers in agreement with the predictions of Massot, Irani, and Lightfoot. This behavior, in contradiction with the ever decreasing wavelength predictions of Kapitza, Benjamin, and Yih, has been observed experimentally by Massot and Irani (*13*) in unpublished work and is apparent from the data of Kapitza (*9*) and Stainthorp and Allen (*16*). However, experimental results seem consistently higher than the theoretical predictions, although predicting the correct trend of the variations. The solutions presented here, however, lie closer to experiment.

Dimensionless Wave Celerity

A comparison between the existing theories and the present results appears in Figure 3 representing the variation of the dimensionless celerity α with Weber number. The present theories approach the limiting value of 3 as the Weber number approaches zero, in agreement with all the previous low Weber number analyses. At medium Weber numbers there is qualitative agreement between the experimental results and the present predictions. The present analysis predicts closer agreement with experimental results than existing linear analyses. The experimental results lie about 25% lower than Massot, Irani, and Lightfoot's treatment for Reynolds numbers less than 100. The departure between the theories and experiments may be due partly to nonlinear effects, since experimental observations of Allen (*16*) indicate the presence of asymmetric waves on the film surface in this regime.

In the treatments considered the dimensionless celerity and wave number are functions of the Weber number only. The fluid properties group is only introduced in determining the dimensionless wavelength in the present treatment, whereas Benjamin's approximation for long waves predicts that α and N_W^2 are functions of both the Weber number and the fluid properties group (Figures 4 and 5). For higher Weber numbers both the results of the second approximation of the present work and the solution of Massot et al. approach the predictions of Ishihara, Iwagaki, and Iwasa. However extrapolation of the results to this region are of analytic interest only and do not have any physical meaning, since for film flow conditions this would be well outside the laminar regime and the range of validity of the theory.

Wave Number

The variation of the dimensionless wave number with the Weber number is illustrated in Figure 5 for the various linear analyses. The first approximation is in agreement with the instability theory analysis of Hanratty and Hershman. The second approximation predicts the existence of a maximum value of the wave number, in contradiction

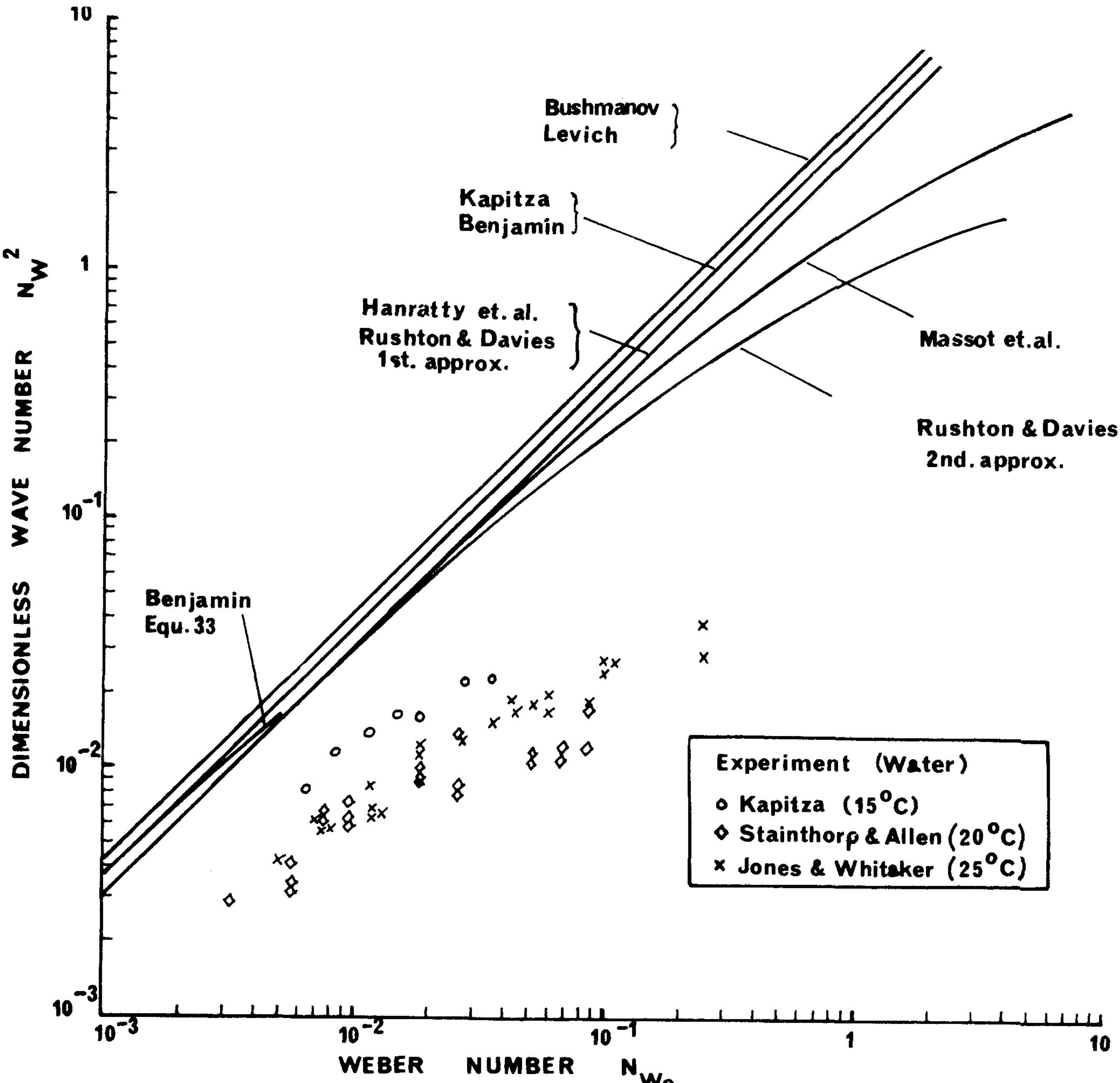

Fig. 5. Dimensionless wave number vs. Weber number—a comparison between theory and experiment.

with the increasing wave number predictions of Kapitza, Benjamin, Yih, and Hanratty and Hershman. This behavior has been observed experimentally by Kapitza and Stainthorp and Allen.

Discussion of the Approximations

The discrepancy between the experimental data and the theoretical predictions for low to medium Weber numbers can be attributed to two factors. First the velocity profile approximation of the Nusselt type provides a good approximation for low Reynolds numbers, but for Reynolds numbers greater than 20 its validity becomes questionable. Experimental velocity profile measurements in the wavy flow regime have been obtained by Wilkes and Nedderman (*18*). Their results are restricted to very low values of the Reynolds number ($N_{Re} < 4$). However the profile,with surface waves present, was distributed about Nusselt's semiparabolic profile. This provides some qualitative agreement and confirmation of the profile assumed in the present analysis. Velocity profile measurements for higher Reynolds numbers within the wavy laminar flow regime would be beneficial and useful for further theoretical analyses.

The second factor affecting the discrepancy between the experimental data and the theoretical predictions is the assumption of linearity. For low Weber numbers, hence long waves, the effects should be small. However on increasing the Weber number the nonlinear effects will become important and influence the results. Hence departures between experiments and the linear theories for medium Weber numbers should not be surprising.

Considering the microscopic structure of the flow predicted from this type of treatment results in the streamlines passing out through the free surface. This cannot have any physical justification and is a result of accepting a semiparabolic velocity profile as an initial assumption. This point emphasizes the need for some detailed experimental examination of the flow structure.

SUMMARY

In this paper a thin film parameter ζ has been defined and an order analysis of the dimensionless equations of motion and boundary conditions in this parameter has been established for vertical two-dimensional flow of thin liquid films. Solutions based on Kapitza's assumption of a semiparabolic velocity profile in terms of a displacement parameter φ are presented which provide closer agreement with experimental results than previously published analyses using this technique. The quantitative agreement with experiment is still poor. This is due to the omission of nonlinear and higher order terms in φ and in restricting the solution to a semiparabolic velocity profile.

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NOTATION

c = wave velocity
g = gravitational acceleration
h = local film thickness
h_0 = mean film thickness
H = h/h_0 = dimensionless film thickness
n = $2\pi/\lambda$ = wave number
N_{fp} = $3\gamma^3/g\nu^4$ = dimensionless fluid properties group
N_{Fr} = $u_0/\sqrt{gh_0}$ = Froude number
N_{Re} = $4h_0u_0/\nu$ = Reynolds number
N_W = nh_0 = dimensionless wave number
N_{We} = $h_0u_0^2/\gamma$ = Weber number
N_λ = $\frac{\lambda}{2\pi}\sqrt{\frac{g}{\gamma}}$
p = pressure
p_0 = ambient pressure
p_σ = capillary pressure
P = $p/\rho u_0^2$ = dimensionless pressure
R = $N_{Re}/4$ = modified Reynolds number
t = time
T = $tu_0\zeta/h_0$ = dimensionless time
u_0 = velocity of mean stream section h_0
u_x = velocity component in x direction
u_y = velocity component in y direction
U = u_x/u_0 = dimensionless velocity
$\overline{U}_1$ = dimensionless mean velocity in X direction (first approximation)
$\overline{U}_2$ = dimensionless mean velocity in X direction (second approximation)
V = $u_y\lambda/(u_0\zeta)$ = dimensionless velocity
x = distance in direction of stream
x_0 = x has a range λ from $x = x_0$ to $x = x_0 + \lambda$
X = $(x - x_0)/\lambda$ = dimensionless distance
y = distance perpendicular to wall
Y = y/h_0 = dimensionless distance

Greek Letters

α = c/u_0 = dimensionless wave celerity
γ = σ/ρ = kinematic surface tension
δ = $\tan^{-1}(\partial h/\partial x)$
ζ = h_0/λ = thin film parameter
θ = amplitude
λ = wavelength
ν = kinematic viscosity
ρ = liquid density
σ = liquid surface tension
φ = $(h - h_0)/h_0$ = dimensionless displacement
$\underline{\varphi}$ = average value of φ per wavelength

Superscripts

$'$ = first derivative with respect to $(X - \alpha T)$
$''$ = second derivative with respect to $(X - \alpha T)$
$'''$ = third derivative with respect to $(X - \alpha T)$

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